One-arm Interferometric Detectors of Gravitational Waves

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Abstract

We discuss interferometric detection of gravitational waves using multiple bounce parallel-hca~l) systems. The design we consider allows us to remove the laser frequency fluctuations, and gives a remaining non-zero gravitational wave signal. The resultant sensitivity, however, is about B times smaller than the sensitivity of a two-arm Michelson interferometer operating with B reflections. A space-based version of our design, sensitive to kilohertz radiation and with an arm length of about 100 km, would require only one reflection. This would make it as sensitive as a right-angled one-bounce Michelson interferometer of similar arm length.

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1 Introduction

Non-resonant detectors of gravitational radiation (with frequency content $0 < f < f_0$) are essentially interferometers with one or more arms, in which a coherent train of electrom agnet ic waves (of nominal frequency $\nu_0 >> f_0$) is folded into several beams, and at points where these intersect relative fluct uations of frequency or phase are monitored (homodyne detection). Frequency flue.trlatiol]s in a narrow band can alternatively be described as fluctuating sideband amplitudes, and interference of two or more beams, produced and mon-itored by a (nonlinear) device such as a photodetector, exhibits these sidebands as a low frequency signal agai n with frequency content $0 < f < f_0$. The observed low frequency signal is due to frequency Variations of the source of the beams about ν_0, to relative motions of the source and the mirrors (or amplifying transponders) that do the folding, to temporal variations of the index of refraction along the beams, and, according to general relativity, to any time-variab le gravitational fields present, such as the transverse traceless metric curvature of a passing plane gravitational wave train. To observe these gravitational fields in this way, it is thus necessary to control, or monitor, the other sources of relative frequency fluctuations, and, in the data analysis, to optionally use algorithms based 011 the different characteristic interferometer responses to gravitational waves (the signal) and to the other sources (the noise). Several feasibility studies [1-4] have shown that this can presently be done to astrophysically interesting thresholds for both ground and sl)acc-based instruments.

The frequency band in whit.11 a ground-based interferometer can be made most sensitive to gravitational waves [2] ranges from about ten Hertz to about a few kiloHertz, with arm lengths ranging from afcw tens of meters to a few kilometers. Spat.c-basccl interferometers, such as the coherent microwave tracking of interplanetary spacecraft [3] and proposed Michelson optical interferometers in planetary orbits [4] arc most sensitive to milliHertz gravitational waves, with arm lengths ranging from 10^6 to 10^8 kilometers.

in present single-spacecraft Doppler tracking observations, many of the noise sources can be either reduced or calibrated by implementing appropriate microwave frequency links and by using specialized hardware, so the fundamental limitation is imposed by the frequency (time-keeping) flue.tuat.iorls inherent to the reference clocks that control the mi crowave system. Hydrogen maser clocks, currently used in Doppler track ing experiments, achieve their best performance at shout 1000 seconds integration time, with a fractional frequency stability of afcw parts in 10^{16}. This is the reason why these one-arm interferometers arc most sensitive to milliHertz gravitational waves. This integration time is also comparable to the storage time L/c for spacecraft en route to the outer solar system ($L \simeq 3AU$), so these interferometers have near-optimum response to gravitational radiation.

By comparing phases of split beams propagated along non-parallel arms [2,4,8,11], source frequency fluctuations can be removed and gravitational wave signals at lower levels can in principle be detected. Especially for interferometers that use light generated by presently available lasers, which have frequency stability roughly a few parts in 10^{-13}, it is essential to be able to remove these fluctuations when searching for gravitational waves of dimensionless amplitude less than 10^{-19} in the milliHertz band [4], down to 10^{-21} - 10^{-23} desired in the kiloHertz frequency band [2]

The usual way of operating an interferometer implies, however, that the frequency side bands induced by a gravitational wave vanish when the arms of the interferometer are parallel. It is interesting therefore to consider whether there exist alternative ways of making measurements with an interferometer of parallel beams. This would be applicable, for instance, to situations in which site constraints would allow the construction of only one long vacuum pipe [5]. The possibility of implementing an interferometer detector with only one arm would also imply that orthogonal-arm vacuum installations would actually be capable of generating two stream of data, from independent one-arm systems. This would provide both redundancy in the data analysis and useful extra directional information about the signal. In this paper we address this problem, and as an example propose a particular design for a multiple bounce one-arm interferometer, in which offset beams are driven by the same laser light.

In Section II we deduce from first principles the response function of a single-arm folded beam to a plane gravitational wave train. In the long wavelength limit (arm length << gravitational wavelength) the usual expression for the phase shift of a many bounce system [6] is recovered. In Section III, after describing our proposed design of a one-arm interferometer, we deduce its response function to a plane gravitational wave train. The data from such an interferometer give a non-zero gravitational wave signal, and a remaining laser phase noise of magnitude smaller than the signal, although the usual advantage of having many bounces is lost. Our method also relies on the capability of independently measuring the lengths of the arms; in Section IV we deduce an analytic expression for the necessary precision. For presently available lasers, in order to reduce their noise to a level of 10--2], the corresponding relative precision in arm length must be less than 10^{-8}, which should be achievable with auxiliary ranging or by taking advantage of the "two pulse" time dependence of the laser noise itself [7] Finally in Section V we present ou - comments and conclusions.

II The Response Function for a folded beam

The net effect of a weak gravitational wave train on the frequency of a coherent light beam reflected once in a stationary, freely falling, configuration of source and mirror is the so called *three-pulse response function* [1,7,8]. A gravitational wave pulse contributes to the interferometrically measured phase shift at three times, namely at the time it is incident on the source, the intermediate time when the light bounces off the end mirror, and at the round-trip light time.

In this Section we will deduce the general expression for the phase shift due to a gravitational wave when the laser light is made to bounce B times between two freely falling (geodesic) mirrors of very high reflectivity. The source of the light is at the first mirror, and the net frequency change, or equivalent phase fluctuation, is interferometrically measured there.

Let us consider the spat.c-tilnc metric

$$ds^2 = --(112-1\ (1 + h)dx^2 + (1 - h)dy^2 + dz^2 \,, \tag{1}$$

where $h = h(t - z) << 1$. To first order, this is tllc.general relativistic solution for the strain field of a linearly polarized gravitational wave train propagating in vacuum along the positive z direction. The metric could be generalized by adding in an amplitude for the other possible polarization) but to first order it is just as easy to do this at the conclusion, as needed. Let us also assume that our two mirrors are stationary in the (x, z) plane. The relative geometry is described in Figure 1; we have denoted by α the cosine of the angle between the direction of propagation of the gravitational wave and the line joining mirror a to mirror b.

In this space-tilllc the mirrors follow' a geodesic. motion, represented by world lines parallel to the t axis. With the geometry described in Figure 1, we can visualize our physical system within the space-time diagram shown in Figure 2. The vertical axis is the time t, while the horizontal axis is the line $\alpha z + \beta x$, where $\beta^2 = 1 - \alpha^2$. The t axis coincide with the world line $x = y = z = 0$ of mirror a, while the world line for mirror b is (to first order in h): $x = \beta L$, $y = 0$, and $z = \alpha L$. The characteristic wave fronts of the gravitational wave are given by $t - z =$ *constant.*

Consider, at an arbitrary time t, a perfectly monochromatic photon of frequency ν_0 (as measured in the rest frame of a) emitted from a laser at a, which bounces off the end mirror b at time $t+L$, and then returns to mirror a at time $t+2L$. In Figure 2 this trajectory is represented by two null geodesics, one originating at the event labelled O and ending at the event 1; the other connects the event 1 to the event 2. Parallel transport of a null vector along these null geodesic is used to calculate ν_1, the frequency measured at event 1 in the rest frame of b, and ν_2 at event 2 again in the rest frame of cf.

The frequency shifts $\nu_1 - \nu_0$, and $\nu_2 - \nu_1$ are related to the gravitational wave amplitude according to the following simple "two- pulse" relationships [i'] (also see Eqs. (13) and (19) of Ref. [1])

$$\frac{\nu_1(t+L)}{\nu_0} = 1 + \frac{(1+\alpha)}{2}\left[h(t) - h(t+(1-\alpha)L)\right], \tag{2}$$

$$\frac{\nu_2(t+2L)}{\nu_1(t+L)} = 1 + \frac{(1-\alpha)}{2}\left[h(t+(1-\alpha)L) - h(t+2L)\right], \tag{3}$$

where ν_0 is independent of time, since for the moment we are considering a monochromatic light source (or "atomic" frequency standard)

If we multiply together Eq. (2) and Eq. (3), and disregard second order terms in the wave amplitude h, we deduce the three-pulse response function in its original form [7]

$$\frac{\nu_2(t+2L)}{\nu_0} = 1 + \frac{(1+\alpha)}{2}\, h(t) - \alpha\, h(t+(1-\alpha)L) - \frac{(1-\alpha)}{2}\, h(t+2L), \tag{4}$$

Eq. (4) is then best rewritten to display the fractional frequency change at a as a function of time t

$$y(t) \equiv \frac{\nu_2(t) - \nu_0}{\nu_0} = -\frac{(1-\alpha)}{2}\, h(t) - \alpha\, h(t-(1+\alpha)L) + \frac{(1+\alpha)}{2}\, h(t-2L). \tag{5}$$

The phase difference $\Delta\phi^{(1)}(t)$ measured, say, by a photo detector is related to the corresponding frequency change, given by Eq. (5), as follows

$$y(t) = \frac{1}{2\pi\nu_0}\frac{d\Delta\phi^{(1)}(t)}{dt}. \tag{6}$$

If we define the Fourier transform of the time series $\Delta\phi^{(1)}(t)$ to be given by

$$\widetilde{\Delta\phi^{(1)}}(f) \equiv \int_{-\infty}^{+\infty} \Delta\phi^{(1)}(t) e^{2\pi i f t} \, dt \ , \tag{7}$$

we can rewrite Eq. (5) in the Fourier domain as

$$\frac{\widetilde{\Delta\phi^{(1)}}(f)}{2\pi\nu_0} = \frac{R(f)}{2\pi i f} \tilde{h}(f). \tag{8}$$

In Eq. (8) $R(f)$ is the three-pulse transfer function

$$R(f) = -\frac{(1-\alpha)}{2} - \alpha e^{2\pi i(1+\alpha)fL} + \frac{(1+\alpha)}{2} e^{4\pi i f L}. \tag{9}$$

For those who prefer to think in terms of heterodyne detection, of signals on a carrier of amplitude A_0 and frequency ν_0, this phase modulation engenders side bands of amplitude A given by

$$\frac{A(\nu_0 + f)}{A_0} = \frac{\nu_0}{f} [R(f) \, R(f)^*]^{1/2} \ \tilde{h}(f). \tag{10}$$

If we expand Eq. (9) in the long wavelength limit ($fL << 1$), to first order in fL Eq. (8) becomes [8]

$$\frac{\widetilde{\Delta\phi^{(1)}}(f)}{2\pi\nu_0} \simeq (\alpha^2 - 1) \ L \ [1 + \pi i(\alpha + 2)fL] \ \tilde{h}(f). \tag{11}$$

The factor $(\alpha^2 - 1)$ is the "beam pattern" of a single-bounce linear gravitational wave antenna. In the long wavelength limit, its "antenna gain" is $\approx L$.

Let us now assume that the light inside the arm makes B bounces before it is made to interfere with the light of the laser. We want to determine what the corresponding phase change will be in this case. From Figure 2 we note that the frequencies $\nu_2(t+2L)$, $\nu_3(t+3L)$, and $\nu_4(t+4L)$, for instance, are related among themselves as ν_0, $\nu_1(t+L)$, and $\nu_2(t+2L)$ assuming proper care of the time argument is taken. We can for example easily find that the following expression for $\nu_4(t+4L)/\nu_2(t+2L)$ holds

$$\frac{\nu_4(t+4L)}{\nu_2(t+2L)} = 1 + \frac{(1+\alpha)}{2} \ h(t+2L) - \alpha \ h(t+2L+(1-\alpha)L)$$

$$- \frac{(1-\alpha)}{2}\, h(t+4L). \tag{12}$$

If we multiply Eq. (4) by Eq. (12) we get, to first order in h,

$$\frac{\nu_4(t+4L)}{\nu_0} \simeq 1 + \frac{(1+\alpha)}{2}\, h(t+2L) - \alpha\, h(t+2L+(1-\alpha)L) - \frac{(1-\alpha)}{2}\, h(t+4L)$$

$$+ \frac{(1+\alpha)}{2}\, h(t) - \alpha\, h(t+(1-\alpha)L) - \frac{(1-\alpha)}{2}\, h(t+2L). \tag{13}$$

If we use the definition of $y(t)$ given in Eq. (5), Eq. (13) can be rewritten in the following way

$$\frac{\nu_4(t) - \nu_0}{\nu_0} = y(t) + y(t-2L). \tag{14}$$

After some simple algebra we can easily deduce the following expression for the frequency change after B bounces

$$\frac{\nu_{2B}(t) - \nu_0}{\nu_0} = \sum_{k=0}^{B-1} y(t-2kL). \tag{15}$$

Let us now denote by $\Delta\phi^{(B)}(t)$ the phase shift measured at the photo detector for the B bounce configuration. Taking into account Eq. (15), we can write the following equation

$$\frac{1}{2\pi\nu_0}\frac{d\Delta\phi^{(B)}(t)}{dt} = \sum_{k=0}^{B-1} y(t-2kL), \tag{16}$$

which in the Fourier domain becomes

$$\frac{\widetilde{\Delta\phi^{(B)}}(f)}{2\pi\nu_0} = -\frac{\widetilde{y}(f)}{2\pi i f} \sum_{k=0}^{B-1} e^{4\pi i k f L}. \tag{17}$$

From the definition of $y(t)$ (Eq. (5)), and after adding the geometric progression, we can rewrite Eq. (17) as

$$\frac{\widetilde{\Delta\phi^{(B)}}(f)}{2\pi\nu_0} = \frac{R(f)\,\widetilde{h}(f)}{2\pi i f}\left[\frac{1 - e^{4\pi i B f L}}{1 - e^{4\pi i f L}}\right] \tag{18}$$

If we expand Eq. (18) in the long wavelength limit, that is to say when $fL \ll 1$ but allowing B to be large enough that $4BfL \simeq 1$, for the dominant frequency band of the gravitational wave signal, we get

$$\frac{\widetilde{\Delta\phi^{(B)}}(f)}{2\pi\nu_0} \simeq \frac{(1-\alpha^2)}{2} \frac{(1-e^{4\pi iBfL})}{2\pi if} [1 + \pi i(\alpha+2)fL]\, \tilde{h}(f) \tag{19}$$

Note that the transfer function given in Eq. (19) dots not increase linearly with the arm length, as it did for the one-bounce configuration, $B = 1$. For a given arm length L and for a gravitational wave signal of dominant frequency f, one can choose the number of reflections B in such a way that $4BfL \simeq 1$, and the response is optimal, depending only on j and the geometrical factor $(1 - \alpha^2)$.

Note that this condition also holds for a Michelson interferometer, since its transfer function is essentially equal to the one given in Eq. (19), apart from a different antenna pattern [9,10]. At one kiloHertz an orthogonal-arm interferometer, of 40 meters arm length and $B \simeq 2000$ bounces, would experience the same phase shift due to a passing gravitational wave as would an interferometer of' 4 kilometer arm length and $B \simeq 20$ bounces.

1 I 1 Offset parallel arms

Let us consider the optical configuration described in Figure 3. We have two parallel folded beams disposed sequentially (one after the other), each of total length $2BL$, but offset by a distance 1. This setup will be referred to as (offset) parallel arms. In Figure 3 the distance l has been assumed, for sake of clarity, to be larger than L. At an arbitrary time t a perfectly monochromatic laser light of frequency ν_0 is injected into the first arm. It bounces inside the arm B times, and then part of the light is made to interfere with the incoming beam, while the remaining light is fed into the next arm. The light that enters in the second arm also makes B bounces, and then interferes with the light shining on its input port. That is, two phase differences are measured at the same time, at tile two offset input ports.

This physical configuration is represented by the space-time diagram given in Figure 4. Here we have four world lines for the four mirrors. As in Figure 2, the world lines of mirrors a and b are given by $x = y = z = 0$, and $x = \beta L$, $y = 0$, $z = \alpha L$ respectively. The world lines of mirrors c and d are given (to first order in h) by the equation $x = \beta l$, $y = 0$, $z = \alpha l$ and $x = \beta(l + L)$, $y = 0$, $z = \alpha(l + L)$ respectively. The trajectory of the light is represented by $(4B + 1)$ null geodesics. The first $2B$ null geodesics connect, sequentially, events on the time-like geodesics of mirrors a and b. A null geodesic connects the event labelled $2B$ to the event labelled 1, which represent the event at mirror c when the light enters the second arm. Finally the remaining $2B$ null geodesics connect events that are located on the Li]lm-like geodesics of mirrors c and d.

The quantity measured interferometrically in the second arm is the relative frequency change $[\nu_{4B+1}(t) - \nu_1(t)]/\nu_1(t)$. In what follows we will deduce its dependence on the gravitational wave amplitude. Let us first rewrite the response of the first arm [Eq. (1 5)] in the following form

$$\frac{\nu_{2B}(t+2BL)}{\nu_0} = 1 + \sum_{k=0}^{B-1} y(t + 2BL - 2kL). \tag{20}$$

The frequency at the event 1, $\nu_1(t+2BL+l)$, is related to the frequency $\nu_{2B}(t+2BL)$ by the following relationship (see Eq. (2))

$$\frac{\nu_1(t+2BL+l)}{\nu_{2B}(t+2BL)} = 1 + \frac{(1+\alpha)}{2}\left[h(t+2BL) - h(t+2BL+(1-\alpha)l)\right] \tag{21}$$

For the remaining $2B$ bounces in the second arm, one can easily deduce the following equation

$$\frac{\nu_{4B+1}(t+4BL+l)}{\nu_1(t+2BL+l)} = 1 + \sum_{k=0}^{B-1} y(t + 4BL + (1-\alpha)l - 2kL). \tag{22}$$

If we multiply Eq (22) by Eqs. (21), (20), to first order in h we get the following expression for $\nu_{4B+1}(t + 4BL + l)/\nu_0$

$$\frac{\nu_{4B+1}(t+4BL+l)}{\nu_0} = 1 + \sum_{k=0}^{B-1} y(t + 2BL - 2kL) + \sum_{k=0}^{B-1} y(t + 4BL + (1-\alpha)l - 2kL)$$

$$+ \frac{(1+\alpha)}{2}\left[h(t+2BL) - h(t+2BL+(1-\alpha)l)\right]. \tag{23}$$

If we multiply Eq. (2?1) by Eq. (20) we get

$$\frac{\nu_1(t+2BL+l)}{\nu_0} = 1 + \frac{(1+\alpha)}{2}\left[h(t+2BL) - h(t+2BL+(1-\alpha)l)\right]$$

$$+ \sum_{k=0}^{B-1} y(t + 2BL - 2kL). \tag{24}$$

So Eq. (23) can be rewritten as

$$
\frac{\nu_{4B+I}(t)}{\nu_0} = 1 + \sum_{k=0}^{B-1} y(t - 2BL - l - 2kL) + \sum_{k=0}^{B-1} y(t - \alpha l - 2kL)
$$

$$
+ \frac{(1+\alpha)}{2} \left[h(t - 2BL - l) - h(t - 2BL - \alpha l)\right] , \tag{25}
$$

while Eq. (24) becomes

$$
\frac{\nu_I(t)}{\nu_0} = 1 + \sum_{k=0}^{B-1} y(t - 2kL - l) + \frac{(1+\alpha)}{2} \left[h(t - l) - h(t - \alpha l)\right] \tag{26}
$$

By dividing Eq. (25) by Eq. (26) we get, to first order in h

$$
\frac{\nu_{4B+I}(t)}{\nu_I(t)} = 1 + \sum_{k=0}^{B-1} y(t - 2BL - 2kL - l) + \sum_{k=0}^{B-1} y(t - 2kL - \alpha l) - \sum_{k=0}^{B-1} y(t - 2kL - l)
$$

$$
+ \frac{(1+\alpha)}{2} \left[h(t - 2BL - l) - h(t - 2BL - \alpha l) - h(t - l) + h(t - \alpha l)\right] , \tag{27}
$$

We note that, to first order in h, the relative frequency change $[\nu_{4B+I}(t) - \nu_I(t)]/\nu_I(t)$ is equal to $[\nu_{4B+I}(t) - \nu_I(t)]/\nu_0$. This allows us to express the relative frequency changes, given in Eqs. (15),(27), in terms of the phase changes defined below

$$
\frac{1}{2\pi\nu_0} \frac{d\Delta\phi_1(t)}{dt} = \frac{\nu_{2B}(t) - \nu_0}{\nu_0} \tag{28}
$$

$$
\frac{1}{2\pi\nu_0} \frac{d\Delta\phi_2(t)}{dt} = \frac{\nu_{4B+I}(t) - \nu_I(t)}{\nu_0} . \tag{29}
$$

In Eqs. (15), (27) derived above we considered only the effect of a gravitational wave on the measured phase shift, If we also take into account the phase shifts due to the laser fluctuations, and those due to the possible uncorrelated remaining noise sources affecting the output of the one-arm response, the relative frequency differences (Eqs. (28), (29)) assume the following form

$$
\frac{1}{2\pi\nu_0} \frac{d\Delta\phi_1(t)}{dt} = \sum_{k=0}^{B-1} y(t - 2kL) - \frac{1}{2\pi\nu_0} \left[\frac{dP(t - 2BL)}{dt} - \frac{dP(t)}{dt}\right] + \frac{1}{2\pi\nu_0} \frac{dn_1(t)}{dt} , \tag{30}
$$

$$\frac{1}{2\pi\nu_0}\frac{d\Delta\phi_2(t)}{dt} = \sum_{k=0}^{B-1} y(t-2BL-2kL-l) + \sum_{k=0}^{B-1} y(t-2kL-\alpha l) - \sum_{k=0}^{B-1} y(t-2kL-l)$$

$$- \frac{(1+\alpha)}{2}\left[h(t-2BL-l) - h(t-2BL-\alpha l) - h(t-l) + h(t-\alpha l)\right]$$

$$+ \frac{1}{2\pi\nu_0}\left[\frac{dP(t-4BL-l)}{dt} - \frac{dP(t-2BL-l)}{dt}\right] + \frac{1}{2\pi\nu_0}\frac{dn_2(t)}{dt}, \tag{31}$$

where we have denoted by $P(t)$ the laser phase noise, and by $n_i(t)$; ($i = 1, 2$) the phase change due to the remaining noise sources. It can be argued that most of these noise terms are inversely proportional to the arm length L [6] implying that the longer the arm length, the smaller the contribution to the overall phase change due to these terms.

Of all the noise sources, the laser frequency noise is the largest, being eight to ten orders of magnitude larger than the amplitude of any other noise source [2]. In a regular Michelson Interferometer the laser phase fluctuations propagate along the two orthogonal and almost equal length arms, and when the returning beams are recombined after each makes B bounces, the fluctuations are delayed by equal times and so cancel. Directly, without need for independent readouts, the gravitational wave signal of course will not cancel out since each arm is affected differently by the wave due to its transverse traceless nature, and its response function can be optimized. In our parallel arm optical configuration, we also want to combine linearly the two data sets, measured at the output ports of the two arms, in such a way to cancel the laser noise and still retain the gravitational wave signal. As we shall show below, this can be done; the magnitude of the remaining gravitational wave signal, however, is about B times smaller than what one would detect with a two-orthogonal-arm Michelson Interferometer.

Let us assume, for the moment, that the arm length L and the offset distance l are known exactly. From Eqs. (30), (31) we note that, by time shifting the data set from the second arm with respect to those of the first arm by $2BL + l$, and then subtracting them from the data of the first arm, we remove the laser noise. The so formed new frequency change, $d\Delta\phi_p(t)/dt$, contains the following terms

$$\frac{1}{2\pi\nu_0}\frac{d\Delta\phi_p(t)}{dt} = \frac{1}{2\pi\nu_0}\left[\frac{d\Delta\phi_1(t)}{dt} - \frac{d\Delta\phi_2(t+2BL+l)}{dt}\right] = \sum_{k=0}^{B-1}\left[y(t+2BL-2kL)\right.$$

$$\left.- y(t+2BL-2kL+(1-\alpha)l)\right] - \frac{(1+\alpha)}{2}\left[h(t) - h(t+(1-\alpha)l) - h(t+2BL)\right.$$

$$+ h(t + 2BL + (1 - \alpha)l) \Big] + \frac{1}{2\pi\nu_0} \left[\frac{dn_1(t)}{dt} - \frac{dn_2(t + 2BL + l)}{dt} \right] . \qquad (32)$$

After some simple algebra, Eq. *(32)* can equivalently be written as a transfer function, in the Fourier domain

$$\frac{1}{2\pi\nu_0} \widetilde{\Delta\phi_p}(f) = \left[\frac{R(f)}{1 - e^{4\pi i f L}} + \frac{(1 + \alpha)}{2} \right] \frac{\left(1 - e^{2\pi i(\alpha - 1)fl}\right)}{2\pi i f} \left(1 - e^{-4\pi i B f L}\right) \widetilde{h}(f)$$

$$+ \frac{1}{2\pi\nu_0} \left[n_1(f) - n_2(f) e^{-2\pi i f(2BL + l)} \right] , \qquad (33)$$

where $R(f)$ is the three-pulse transfer function given in Eq. *(9)*.

If we expand again Eq. *(33)* in the long wavelength limit ($fL << 1$ but $4BfL \simeq 1$), for the dominant frequency band of the gravitational wave signal, we get

$$\frac{1}{2\pi\nu_0} \widetilde{\Delta\phi_p}(f) \simeq \frac{\alpha(\alpha^2 - 1)}{2} \, l \, \left(1 - e^{-4\pi i B f L}\right) \widetilde{h}(f) + \frac{1}{2\pi\nu_0} \left[n_1(f) - n_2(f) e^{-2\pi i f(2BL + l)} \right] . \qquad (34)$$

Eq. (34) shows some interesting, and somewhat peculiar properties of the remaining gravitational wave signal. First of all we note that when the offset l is equal to zero the gravitational wave signal also vanishes. This is a general result, valid for any α and any gravitational wavelength (see Eq. (33) above). In fact, when $l = 0$ the gravitational wave signal in the second arm is delayed by the same amount as the laser noise, and therefore when we remove the laser noise we also remove the wave signal. "This delay effect" explains also the unusual antenna pattern, or dependence on α, deduced in Eq. (34). The transverse gravitational wave signal goes to zero not only when the wave propagates along the direction of the arms ($\alpha = \pm 1$), but also when it propagates orthogonally to the arms themselves ($\alpha = 0$). For $\alpha = 0$ the "three-pulse" response of any one-arm interferometer, Eq. (19), becomes a "two-pulse" response identical to that for a laser fluctuation in Eq. (30), and therefore the two gravitational wave signals in the combined data set will cancel out. We finally note that the maximum value of the antenna pattern given in Eq. (34) is equal to $\sqrt{3}/9$, while for a regular Michelson interferometer the maximum is equal to 1. This allows us to compare, for each Fourier component of the same wave amplitude h, the maximum value of the phase shift $\widetilde{\Delta\phi_p^h}(f)$ induced by a wave in a parallel-arm interferometer against the corresponding one, $\widetilde{\Delta\phi_m^h}(f)$, experienced by a Michelson interferometer. We find the following ratio of the two phase shifts at an arbitrary Fourier frequency f

$$\frac{\widetilde{\Delta\phi^h_p}(f)}{\widetilde{\Delta\phi^h_m}(f)} \simeq 1.3 \times i f l. \tag{35}$$

For a gravitational wave signal of dominant frequency 1 kHz, and assuming l to be about 2 km, an offset parallel-arm interferometer would observe a gravitational wave effect 100 times smaller than what would be observed by a regular Michelson Interferometer. If the number of bounces B are chosen to maximize the signal at this frequency, than Eq. (35) can be rewritten in the following form

$$\frac{\widetilde{\Delta\phi^h_p}(f)}{\widetilde{\Delta\phi^h_m}(f)} \simeq 1.3 \times \frac{i}{4B}. \tag{36}$$

IV Magnitude of the remaining laser noise

In the previous section the response function of the offset parallel-arm interferometer was derived under the assumption of knowing the length $2BL + l$ exactly. Our knowledge of this length is, however, not exact, being limited by the error we make in measuring it. Let us denote by δ such an error, and let us also assume that terms of the order $\dot{h}(t)\delta/l, \dot{n}_{1,2}(t)\delta/l$, where the dot denotes the time derivative, are negligible with respect to terms of order $h(t)$, and $n_{1,2}(t)$. As we shall show below the precision required in the determination of the arm length justifies this assumption. If we go back to Eqs. (30), (31), and time shift the data set from the second arm with respect to those of the first arm by $2BL + l + \delta$, and then subtract them again from the data of the first arm, we get the following result

$$\begin{aligned}\frac{1}{2\pi\nu_0}\frac{d\Delta\phi_p(t)}{dt} &= \frac{1}{2\pi\nu_0}\left[\frac{d\Delta\phi_1(t)}{dt} - \frac{d\Delta\phi_2(t+2BL+l+\delta)}{dt}\right] \simeq \sum_{k=0}^{B-1}[y(t+2BL-2kL) \\ &\quad - y(t+2BL-2kL+(1-\alpha)l)] + \frac{(1+\alpha)}{2}[h(t) - h(t+(1-\alpha)l) - h(t+2BL) \\ &\quad + h(t+2BL+(1-\alpha)l)] + \left[\frac{1}{2\pi\nu_0}\frac{dn_1(t)}{dt} - \frac{dn_2(t+2BL+l)}{dt}\right] \\ &\quad + \frac{1}{2\pi\nu_0}\left[\frac{d^2P(t)}{dt^2} - \frac{d^2P(t-2BL-l)}{dt^2}\right]\delta. \end{aligned} \tag{37}$$

In the Fourier domain and in the long wavelength limit, for the dominant frequency band of the gravitational wave signal, we get

$$\frac{1}{2\pi\nu_0}\widetilde{\Delta\phi_p}(f) \simeq \frac{\alpha(\alpha^2-1)}{2}\tilde{h}(f)\, l\,(1-e^{-4\pi i BfL}) + \frac{1}{2\pi\nu_0}\left[n_1(f) - n_2(f)e^{-2\pi i f(2BL+l)}\right]$$

$$+\frac{if}{\nu_0}\tilde{P}(f)\,\delta\,\left[e^{2\pi i f(2BL+l)} - 1\right]. \qquad (3s)$$

The equation above allows us 10 estimate the relative precision δ/l required to reduce the laser noise to a level smaller than the gravitational wave amplitude. If we compare the first and the last, terms on the right-hand-side of Eq. (38), we find that the following relationship must hold

$$|\frac{f}{\nu_0}\tilde{P}(f)\frac{\delta}{l}| \ll |\frac{\alpha(\alpha^2-1)}{2}\tilde{h}(f)|. \qquad (39)$$

If we assume a fractional frequency noise due to the laser of 10^{-13} when searching for a signal with typical wave amplitude of 10^{-21}, a separation distance l of about 2 km, and after taking a root-mean-squared value of the antenna pattern over the sphere, Eq. 39 implies the following precision δ in measuring the distance $2BL+l$, required for us to cancel the laser noise to the required level

$$\delta << 2.2 \times 10^{-4} \text{cm}. \qquad (40)$$

V Conclusions

We have discussed a method of interferometric detection of gravitational waves using multiple bounce parallel-beam systems. The main result of our analysis, deduced in Eq. (34), shows that it is possible to remove laser frequency flue.tuatiolls from an offset parallel-beam interferometer without, removing the gravitational wave signal. The magnitude of the remaining gravitational wave's phase shift is, however, about B times smaller than what a regular two-arm Michelson interferometer with B bounces would measure.

In addition to the parallel-bcaln interferometer considered in this paper, we also analyzed several other optics configurations. Although we could not find any improvement with respect to that deduced here, we do not exclude *a priori* the existence of a better and cleverer design that would make parallel-beam interferometry more effective for Earth based detectors. This would be applicable to situations in which site constraints do not allow the construction of two long vacuum pipes along orthogonal directions. It would also imply that regular orthogonal-arm vacuum installations could generate two stream of data, from independent one-arm systems, providing both redundancy in the data analysis and useful directional information about the gravitational wave signal.

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Figure Captions

Figure 1.

Laser light of nominal frequency ν_0 is injected inside two highly reflecting mirrors, a and b. It bounces B times against mirror b, and then is made to interfere with the incoming light from the laser. The gravitational wave train propagates along the z direction, and the cosine of the angle between its direction of propagation and the laser light is denoted by α.

Figure 2.

Space-time diagram describing the optical configuration discussed in Fig. 1. The vertical axis is the time axis t, while the horizontal axis is the line $\alpha z + \beta x$. α is tile cosine of the angle between the direction of propagation of the gravitational wave and the direction of the light; β is determined by the relation $\beta^2 = 1 - \alpha^2$. The geodesic world line of mirror a coincides with the time axis t, while the world line of mirror b is given by $x = \beta L$, $y = 0$, $z = \alpha L$.

Figure 3.

Two parallel folded beams disposed sequentially, each of total length $2BL$, and offset by a distance l. Laser light of frequency ν_0 is injected into first arm. After making B bounces, part of it feeds the next arm, while the remaining light interferes with light from the laser. Light in the second arm also makes B bounces and then interferes with the light shining on its input port.

Figure 4.

Space-time diagram describing the optical configuration discussed in Fig. 3. The vertical axis is the time axis t, while the horizontal axis is the line $\alpha z + \beta x$. α is the cosine of the angle between the direction of propagation of the gravitational wave and the direction of the light; β is determined by the relation $\beta^2 = 1 - \alpha^2$. The world line of mirror a coincide with the time axis t, while the world line of mirror b is given by $x = \beta L$, $y = 0$, $z = \alpha L$. Mirrors c and d are respectively represented by the following world lines: $x = \beta l$, $y = 0$, $z = \alpha l$; $x = \beta(l + L)$, $y = 0$, $z = \alpha(l + L)$.